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A note on second-order perturbations of non-canonical scalar fields

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We study second-order perturbations for a general non-canonical scalar field, minimally coupled to gravity, on the unperturbed FRW background, where metric fluctuations are neglected *a priori*. By employing different approaches to cosmological perturbation theory, we show that, even in this simplified set-up, the second-order perturbations to the the stress tensor, the energy density and the pressure display potential instabilities, which are not present at linear order. The conditions on the Lagrangian under which these instabilities take place are provided. We also discuss briefly the significance of our analysis in light of the possible linearization instability of these fields about the FRW background.

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I. INTRODUCTION

The physics of cosmological perturbations is a well-researched field of study [1–16]. At linear order, the (quantized) cosmological perturbation theory has been the primary tool to investigate the behavior of fluctuations during inflation. For instance, during the slow-roll phase, the linear perturbation theory predicts an approximately scale-free spectrum of density fluctuations, which is consistent with Cosmic Microwave Background (CMB) measurements [17].

During the last few years, considerable amount of attention has been devoted to examine the effect of higher-order corrections to the linearized Einstein's equations. There are three main reasons which have led to such an enormous interest. First, the study of higher-order perturbations is imperative to quantify the primordial non-Gaussanity of the CMB [13, 18–20], which will be confronted with the data collected by the PLANCK mission [21]. Second, within the linear theory, it is not possible to determine when the perturbations become large and non-linearities should be taken into account. For instance, gravitational waves in Minkowski space-time can have arbitrary amplitudes in the linear perturbation theory. The only way to understand the extent of the backreaction of the perturbations on the Friedman-Robertson-Walker (FRW) background is therefore to consider at least the second order [12, 22, 23]. Third, higher-order corrections may help to explain the dark energy. For example, there has been a radical proposal to abandon the Copernican principle and, instead, suppose that we are near the center of a large, non-linearly under-dense, nearly spherical void surrounded by a flat, matter dominated space-time (For recent reviews, see Refs. [24, 25]).

There are four different approaches in the literature to study cosmological perturbations:

1) solving Einstein's equations order-by-order [1];

2) the covariant approach based on a general frame vector u^α [2, 7–9, 12];

3) the Arnowitt-Deser-Misner (ADM) approach based on the normal frame vector n^α [3, 6, 10, 11];

4) the reduced action approach [4, 5].

In the case of linear perturbations, it has been shown that all of these four approaches lead to identical equations of motion. However, to our knowledge, a complete analysis has not been done in the literature for higher-order perturbations (for an earlier study on the differences between the approaches 1) and 3) above, see Ref. [26]). This may be attributed to the following reasons: (a) unlike the linear order, the scalar, vector and tensor perturbations do not decouple and can not be treated independently and (b) although certain physical quantities are gauge-invariant (like tensor metric perturbations) at first order, they may become gauge-dependent at second order [18]. Hence, to obtain gauge-invariant definitions of physically relevant quantities at second (or higher) order is far more complicated [15]. This leads to certain observables having completely different values in different frames.

In this note, to illustrate the problems that may occur at higher-order, and not to get bogged-down with the gauge issues, we consider a simple situation: we freeze all the metric perturbations and focus on the perturbations of a minimally-coupled, generalized scalar field ϕ, whose Lagrangian density is given by [27]

$$\mathcal{L} = P(X,\phi) \ , \qquad \text{where} \qquad 2\,X = \nabla^\alpha\phi\,\nabla_\alpha\phi \ . \qquad (1)$$

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More precisely, we will only consider linear perturbations of the scalar field,

$$\phi(t, \mathbf{x}) = \phi_{0}(t) + \delta\phi(t, \mathbf{x}) \ , \tag{2}$$

about the four-dimensional FRW background,

$$\mathrm{d}s^2 = N^2 \,\mathrm{d}t^2 - \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\mathbf{x}^2 \ , \tag{3}$$

while expanding all the dependent quantities, like X and the stress tensor, up to second order [48]. We again wish to emphasize that our aim is to highlight ambiguities which occur at second order, and not to solve the second-order Einstein equations. In particular, we shall obtain second-order quantities of physical relevance from the scalar field Lagrangian (1), in this simplified set-up, by employing different approaches and highlight the main differences.

To exemplify such differences, it will appear convenient to compare the ratio

$$c_{\mathrm{s}}^2 = \frac{\text{coefficients of } (\delta\phi_{,i}/a)^2}{\text{coefficients of } \delta\dot{\phi}^2} \ , \tag{4}$$

in the components of the stress tensor and related quantities [see Eqs. (15), (23) and (32) below]. Since c_{s}^2 is dimensionally the square of a speed, we will refer to this ratio as the "speed of propagation". We will also discuss the relation between these c_{s}^2 and the square of the phase velocity $c_{\delta\phi}^2$ derived from the wave equation for the perturbation $\delta\phi$ [for small perturbations, Eq. (4) coincides with the standard definition in the theory of elasticity [28]] and of the adiabatic speed of sound $c_{\mathrm{AD}}^2 = \partial p/\partial\rho$, with p the pressure and ρ the energy density. We will then show that the second-order stress tensor $\delta^{(2)}T_{00}$ and the second-order canonical Hamiltonian obtained from the reduced action, which coincide for a canonical scalar field, are in general different.

Among our results, the one which seems of major physical concern will be the emergence of instabilities in the quantities analysed. For example, we shall show that the second-order energy density $\delta^{(2)}\rho$, as defined in the covariant approach, takes a negative contribution from the spatial derivative term, which is also present for the canonical scalar field. Similar terms, which lead to large negative contributions for short wave-lengths, generically appear in the components of the second-order stress tensor and are signalled by imaginary speeds of propagation, that is $c_{\mathrm{s}}^2 < 0$. This finding poses a serious question about the applicability of the perturbative approach (at least) for non-linear scalar field Lagrangians of the form (1).

Our metric signature is $(+,-,-,-)$ and lower case Greek (Latin) indices refer to four (three) dimensions. Time derivatives are denoted by a dot and, for any variable $G(X,\phi)$, its background unperturbed value is denoted by $G^{(0)}$, that is $G^{(0)} \equiv G(X(\phi_0),\phi_0)$. Also, for any scalar function f, we will be using the notations $f_{,i}^2 \equiv \sum_{i=1}^{3}(f_{,i})^2$ and $f_{,ii} \equiv \sum_{i=1}^{3} f_{,ii}$ throughout the paper.

II. EQUATION OF MOTION AND STRESS TENSOR

The equation of motion can be easily derived from the action principle for the general Lagrangian (1),

$$P_X\,\nabla_\mu\nabla^\mu\phi + (\nabla_\mu P_X)\,\nabla^\mu\phi - P_\phi = 0 \ , \tag{5}$$

and the corresponding stress tensor is given by

$$T_{\mu\nu} = P_X\,\nabla_\mu\phi\,\nabla_\nu\phi - g_{\mu\nu}\,P \ , \tag{6}$$

where $P_X \equiv \partial P/\partial X$, $P_\phi \equiv \partial P/\partial\phi$ and so on. Eqs. (5) and (6) are manifestly covariant and hold in arbitrary space-times. We then set out to perturb both expressions up to second order, neglecting the metric perturbations. For instance, the most general perturbation of the stress tensor to all orders can be written as

$$\begin{aligned} T_{\mu\nu} = {}& P_X\Big(X^{(0)} + \Delta X, \phi_0 + \Delta\phi\Big) \\ & \times\nabla_\mu(\phi_0 + \Delta\phi)\,\nabla_\nu(\phi_0 + \Delta\phi) \\ & - \Big(g_{\mu\nu}^{(0)} + \Delta g_{\mu\nu}\Big)\,P\Big(X^{(0)} + \Delta X, \phi_0 + \Delta\phi\Big) \ , \end{aligned} \tag{7}$$

where $\Delta\phi$, ΔX and $\Delta g_{\mu\nu}$ represent perturbations to all orders. We then freeze the metric perturbations by setting $\Delta g_{\mu\nu} = 0$ and expand the other terms up to second order using the form in Eq. (2).

A. Perturbed equation of motion

We start from the equation of motion. The evolution for the background scalar field ϕ_0 is determined by

$$\Big(P_X^{(0)} + P_{XX}^{(0)}\dot{\phi}_0^2\Big)\,\ddot{\phi}_0 + P_{X\phi}^{(0)}\dot{\phi}_0^2 + 3\,\frac{\dot{a}}{a}\,P_X^{(0)}\,\dot{\phi}_0 - P_\phi^{(0)} = 0 \ , \tag{8}$$

while the dynamics of the perturbation $\delta\phi$ is governed by

$$\Big(P_X^{(0)} + P_{XX}^{(0)}\dot{\phi}_0^2\Big)\,\delta\ddot{\phi} - \frac{P_X^{(0)}}{a^2}\,\delta\phi_{,ii} + C\,\delta\dot{\phi} + D\,\delta\phi = 0 \ , \tag{9a}$$

with

$$\begin{aligned} C = {}& 3\,\frac{\dot{a}}{a}\Big(P_X^{(0)} + P_{XX}^{(0)}\dot{\phi}_0^2\Big) + \Big(3\,P_{XX}^{(0)}\ddot{\phi}_0 + P_{X\phi}\Big)\,\dot{\phi}_0 + P_{XX}^{(0)} \\ & + \Big(P_{XX\phi}^{(0)} + P_{XXX}^{(0)}\ddot{\phi}_0\Big)\,\dot{\phi}_0^3 \end{aligned} \tag{9b}$$

$$\begin{aligned} D = {}& 3\,\frac{\dot{a}}{a}\,P_{X\phi}^{(0)}\dot{\phi}_0 + \Big(P_{X\phi\phi}^{(0)} + P_{XX\phi}^{(0)}\ddot{\phi}_0\Big)\,\dot{\phi}_0^2 \\ & + P_{X\phi}\ddot{\phi}_0 - P_{\phi\phi}^{(0)} \ . \end{aligned} \tag{9c}$$

This is a wave equation from which we can immediately read off the speed of propagation of the field $\delta\phi$, namely

$$c_{\delta\phi}^2 = \frac{P_X^{(0)}}{P_X^{(0)} + P_{XX}^{(0)}\dot{\phi}_0^2} \ , \tag{10}$$

which matches the speed of sound used in most of the literature (see, e.g. Ref. [29]). When $c^2_{\delta\phi} < 0$, we therefore expect that the dynamics becomes unstable (see, e.g. Ref. [27]). The nature of this instability is easily understood using an analogy with classical mechanics: when $c^2_{\delta\phi}$ is negative, the system resembles an inverted harmonic oscillator and, no matter how small the amount of perturbation $\delta\phi$, it will rapidly run away from the background solution $\phi_{_0}$ and from the perturbative regime.

Note also that the term $C\,\delta\dot\phi$ in Eq. (9a) would make the frequency ω_k of Fourier modes $\delta\phi_k$ complex, and the perturbation $\delta\phi_k$ would thus decay or grow exponentially in the proper time t. However, terms containing $\delta\dot\phi$ can be eliminated by rescaling

$$\delta\phi(t,\mathbf{x}) \to r(t)\,\chi(t,\mathbf{x}) \ , \tag{11}$$

where r is a suitable function of the background quantities, and this kind of behaviour can thus be studied within the perturbative approach. But this procedure does not change the speed of propagation (10) [nor those defined according to Eq. (4), see Appendix A)] and cannot remove the associated instabilities. In the following, we shall therefore focus only on the instabilities signalled by imaginary speeds of propagation.

B. Perturbed stress tensor

We now turn our attention to the stress tensor and expand it to second order,

$$T_{\mu\nu} = T^{^{(0)}}_{\mu\nu} + \delta^{^{(1)}}T_{\mu\nu} + \delta^{^{(2)}}T_{\mu\nu} \ , \tag{12}$$

with

$$T^{^{(0)}}_{\mu\nu} = P^{^{(0)}}_{_X}\,\dot\phi^2_{_0}\,\delta^0_\mu\,\delta^0_\nu - g_{\mu\nu}\,P^{^{(0)}} \ . \tag{13}$$

Linear perturbations are then given by

$$\delta^{^{(1)}}T_{00} = \left[P^{^{(0)}}_{_X} + P^{^{(0)}}_{_{XX}}\dot\phi^2_{_0}\right]\dot\phi_{_0}\,\delta\dot\phi - \left[P^{^{(0)}}_{_\phi} - P^{^{(0)}}_{_{X\phi}}\dot\phi^2_{_0}\right]\delta\phi \tag{14a}$$

$$\delta^{^{(1)}}T_{0i} = P^{^{(0)}}_{_X}\,\dot\phi_{_0}\,\delta\phi_{,i} \tag{14b}$$

$$\delta^{^{(1)}}T_{ij} = a^2\left(P^{^{(0)}}_{_X}\,\dot\phi_{_0}\,\delta\dot\phi + P^{^{(0)}}_{_\phi}\,\delta\phi\right)\delta_{ij} \ , \tag{14c}$$

and second-order perturbations by

$$\begin{aligned}\delta^{^{(2)}}T_{00} =& \left(P^{^{(0)}}_{_X} + 4\,P^{^{(0)}}_{_{XX}}\dot\phi^2_{_0} + P^{^{(0)}}_{_{XXX}}\dot\phi^4_{_0}\right)\frac{\delta\dot\phi^2}{2} \\ &+ \left(P^{^{(0)}}_{_X} - P^{^{(0)}}_{_{XX}}\dot\phi^2_{_0}\right)\frac{\delta\phi^2_{,i}}{2\,a^2} - \left(P^{^{(0)}}_{_{\phi\phi}} - P^{^{(0)}}_{_{X\phi\phi}}\dot\phi^2_{_0}\right)\frac{\delta\phi^2}{2} \\ &+ \left(P^{^{(0)}}_{_{X\phi}} + P^{^{(0)}}_{_{XX\phi}}\dot\phi^2_{_0}\right)\dot\phi_{_0}\,\delta\phi\,\delta\dot\phi\end{aligned} \tag{15a}$$

$$\delta^{^{(2)}}T_{0i} = \left(P^{^{(0)}}_{_X} + P^{^{(0)}}_{_{XX}}\dot\phi^2_{_0}\right)\delta\dot\phi\,\delta\phi_{,i} + P^{^{(0)}}_{_{X\phi}}\dot\phi_{_0}\,\delta\phi\,\delta\phi_{,i} \tag{15b}$$

$$\begin{aligned}\delta^{^{(2)}}T_{ij} =& P^{^{(0)}}_{_X}\left(1 - \frac{\delta_{ij}}{2}\right)\delta\phi_{,i}\,\delta\phi_{,j} \\ &+ \delta_{ij}\frac{a^2}{2}\Big[\left(P^{^{(0)}}_{_X} + P^{^{(0)}}_{_{XX}}\dot\phi^2_{_0}\right)\delta\dot\phi^2 \\ &+ 2\,P^{^{(0)}}_{_{X\phi}}\dot\phi_{_0}\,\delta\phi\,\delta\dot\phi + P^{^{(0)}}_{_{\phi\phi}}\,\delta\phi^2\Big] \, .\end{aligned} \tag{15c}$$

We would now like to stress the following points regarding the perturbed stress tensor:
i) the components $\delta^{^{(1)}}T_{\mu\nu}$ are identical to the expressions given in Ref. [29] for the case when the metric perturbations are frozen.
ii) for an arbitrary scalar field Lagrangian, $\delta^{^{(2)}}T_{00}$ may represent an unstable perturbation. Indeed, by expanding the perturbations in Fourier modes (so that $\delta\phi^2_{,i} \sim k^2\,\delta\phi^2_k$), one finds that the ratio between the coefficients of $\delta\phi^2_{,i}$ and $\delta\dot\phi^2$ can in general be negative (and become large for large k and/or small a). The origin of this instability is similar to the one we already discussed with regard to the speed of the perturbation obtained from the equation of motion. We will say more on this point later, by considering specific non-canonical Lagrangians, and only remark here that for the canonical scalar field, i.e. for

$$P = X - V(\phi) \ , \tag{16}$$

this problem is not present, since

$$\delta^{^{(2)}}T^{^{(\mathrm{KG})}}_{00} = \frac{\delta\dot\phi^2}{2} + \frac{\delta\phi^2_{,i}}{2\,a^2} + \frac{V_{\phi\phi}}{2}\,\delta\phi^2 \ . \tag{17}$$

iii) by the same token, we observe that $\delta^{^{(2)}}T_{ii}$ is potentially unstable. In this case it is the ratio between $P^{^{(0)}}_{_X}$ and $P^{^{(0)}}_{_X} + P^{^{(0)}}_{_{XX}}\dot\phi^2_{_0}$ which determines the stability of the system. If this ratio is negative, the second-order pressure perturbations are unstable.
iv) only under very special conditions, most notably for the canonical scalar field, the *effective speed of propagation* of $\delta^{^{(2)}}T_{00}$ and $\delta^{^{(2)}}T_{ii}$ are the same as that in Eq. (10) and equal to unity. Using the definition (4), we can define a speed related with the propagation of energy density perturbations in the background frame from $\delta^{^{(2)}}T_{00}$, that is

$$c^2_0 = \frac{P^{^{(0)}}_{_X} - P^{^{(0)}}_{_{XX}}\dot\phi^2_{_0}}{P^{^{(0)}}_{_X} + 4\,P^{^{(0)}}_{_{XX}}\dot\phi^2_{_0} + P^{^{(0)}}_{_{XXX}}\dot\phi^4_{_0}} \ , \tag{18a}$$

and a speed for momentum perturbations from $\delta^{(2)}T_{ii}$,

$$c_\parallel^2 = c_{\delta\phi}^2 \ , \tag{18b}$$

which may be different for non-canonical scalar fields (due to the non-linearity of the dynamics). One then immediately notes that these velocities become imaginary right in correspondence with the instabilities mentioned previously in *iii)* and *iv)*. Finally, it is important to note that it is $c_\parallel$ which equals the speed of perturbations for non-canonical scalar fields given in the literature [29], and we will elaborate about the importance of this result when we discuss the symmetry reduced action.

III. COVARIANT APPROACH

The covariant approach [2] relies upon the introduction of a family of observers travelling with a time-like four-velocity u^μ. By means of u^μ, all the (geometrical) physical objects and operators are decomposed into invariant parts: the scalars along u^μ and scalars, three-vectors, and projected, symmetric and trace-free tensors orthogonal to u^μ. Einstein's equations are then supplemented by the Ricci identities for u^μ and the Bianchi identities, forming a complete set of first-order differential equations (details can be found in Refs. [7, 8, 12, 14]).

The stress tensor for a general scalar field (6) then takes the perfect fluid form

$$T_{\mu\nu} = (\rho + p)\, u_\mu\, u_\nu - p\, g_{\mu\nu} \ , \tag{19}$$

if the time-like unit vector u_μ is chosen as [30, 31]

$$u_\mu = \frac{\nabla_\mu \phi}{\sqrt{2\,X}} \ . \tag{20}$$

In the above, ρ and p are, respectively, the energy density and pressure along the fluid flow and are given by

$$\rho = T_{\mu\nu}\, u^\mu\, u^\nu \ , \quad \text{and} \quad p = -\frac{1}{3}\, T_{\mu\nu}\, h^{\mu\sigma}\, h^\nu{}_\sigma \ , \tag{21}$$

where $h_{\mu\nu} = g_{\mu\nu} - u_\mu\, u_\nu$ is the metric on a slice of fixed observer's time.

On expanding u_μ, $T_{\mu\nu}$ and $h_{\mu\nu}$ up to second order, we obtain

$$\begin{aligned} \overset{(0)}{\rho} &= \overset{(0)}{T}_{00} \ , \quad \delta^{(1)}\rho = \delta^{(1)}T_{00} \\ \delta^{(2)}\rho &= \delta^{(2)}T_{00} - \frac{1}{P_X^{(0)}} \left(\frac{\delta^{(1)}T_{0i}}{a\,\dot{\phi}_0} \right)^2 \end{aligned} \tag{22a}$$

and

$$\begin{aligned} \overset{(0)}{p} &= \overset{(0)}{P} \ , \quad \delta^{(1)}p = -\frac{\delta^{(1)}T^i{}_i}{3} \\ \delta^{(2)}p &= \frac{1}{3\,a^2} \left[\delta^{(2)}T_{ij}\,\delta^{ij} - \frac{1}{P_X^{(0)}} \left(\frac{\delta^{(1)}T_{0i}}{\dot{\phi}_0} \right)^2 \right] \ . \end{aligned} \tag{22b}$$

Substituting (14) and (15) into the above equations yields the second-order corrections

$$\begin{aligned} \delta^{(2)}\rho &= \left(P_X^{(0)} + 4\, P_{XX}^{(0)}\,\dot{\phi}_0^2 + P_{XXX}^{(0)}\,\dot{\phi}_0^4 \right) \frac{\delta\dot{\phi}^2}{2} \\ &\quad - \left(P_X^{(0)} + P_{XX}^{(0)}\,\dot{\phi}_0^2 \right) \frac{\delta\phi_{,i}^2}{2\,a^2} - \left(P_{\phi\phi}^{(0)} - P_{X\phi\phi}^{(0)}\,\dot{\phi}_0^2 \right) \frac{\delta\phi^2}{2} \\ &\quad + \left(P_{X\phi}^{(0)} + P_{XX\phi}^{(0)}\,\dot{\phi}_0^2 \right) \dot{\phi}_0\, \delta\phi\, \delta\dot{\phi} \end{aligned} \tag{23a}$$

$$\begin{aligned} \delta^{(2)}p &= \frac{1}{2} \left(P_X^{(0)} + P_{XX}^{(0)}\,\dot{\phi}_0^2 \right) \delta\dot{\phi}^2 + P_{X\phi}^{(0)}\,\dot{\phi}_0\, \delta\phi\, \delta\dot{\phi} \\ &\quad - \frac{P_X^{(0)}}{2\,a^2}\, \delta\phi_{,i}^2 + \frac{P_{\phi\phi}^{(0)}}{2}\, \delta\phi^2 \ . \end{aligned} \tag{23b}$$

We would then like to stress the following points regarding the perturbed energy density and pressure:

i) Eqs. (22a) and (22b) show that, up to linear order, the energy density and pressure measured in the fluid frame are identical to the same quantities evaluated along the cosmic time. For example, $\rho = T_{00}\, u^0\, u^0$ up to linear order. At second (and higher) order, however, the energy densities measured by these two different observers are no more equal, suggesting that general relativistic effects appear from the second order on. The results obtained here are in fact similar to the corrections derived in the parameterized post-Newtonian formulation [49].

ii) in the fluid frame, the energy density exhibits the same kind of instability we found in the previous section for $\delta^{(2)}T_{00}$, but this time the problem is present also for the canonical scalar field, because of the negative contribution coming from $\delta^{(1)}T_{0i}$. In fact, substituting the Lagrangian (16) in Eq. (23a) yields

$$\delta^{(2)}\rho^{(\mathrm{KG})} = \frac{\delta\dot{\phi}^2}{2} - \frac{\delta\phi_{,i}^2}{2\,a^2} + \frac{V_{\phi\phi}}{2}\, \delta\phi^2 \tag{24}$$

so that, using again the analogy with classical mechanics, the perturbations turn out to be unstable because of the negative sign of the second term in the right hand side (which dominates over the potential term for small a and, in the Fourier domain, for large wavenumber k). Although the results in the two frames (fluid and background) are related by a Lorentz transformation, the authors could not find a discussion of such an instability in standard textbooks [50].

iii) only under special conditions, the *effective speed of propagation* of the energy density and pressure perturbations are equal (as in the previous Section, this occurs for the canonical scalar field). Using the definition (4), the speed of propagation for density perturbations in the fluid frame turns out to be given by

$$c_\rho^2 = -\frac{P_X^{(0)} + P_{XX}^{(0)}\,\dot{\phi}_0^2}{P_X^{(0)} + 4\, P_{XX}^{(0)}\,\dot{\phi}_0^2 + P_{XXX}^{(0)}\,\dot{\phi}_0^4} \ , \tag{25a}$$

and the velocity of pressure perturbations by

$$c_p^2 = c_\parallel^2 = c_{\delta\phi}^2 \ , \tag{25b}$$

from Eq. (18b), and they are obviously different in general. For completeness, we recall that the *adiabatic speed of sound* (see, e.g. Ref. [32]) is given by

$$c^2_{\rm AD} = \left.\frac{\partial p}{\partial \rho}\right|_S = \frac{P^{(0)}_X \ddot{\phi}_0 + P^{(0)}_\phi}{P^{(0)}_X \ddot{\phi}_0 - P^{(0)}_\phi + P^{(0)}_{XX} \dot{\phi}_0^2 \ddot{\phi}_0 + P^{(0)}_{X\phi} \dot{\phi}_0^2} \,, \tag{26}$$

and differs from the other expressions shown so far, and in particular from [32]

$$c^2_{\rm s} = \frac{p_X}{\rho_X} = c^2_{\delta\phi} \ . \tag{27}$$

IV. ADM APPROACH

In the ADM formulation [33], the Einstein-Hilbert action with matter can be written as

$$S = \int {\rm d}t\, {\rm d}^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i - N\, H - N_i\, H^i\right) \ , \tag{28}$$

where N and N^i are the lapse and shift functions, respectively, and π^0 and π^i their conjugate momenta. The super-Hamiltonian and super-momenta are given by

$$H = -\frac{\partial S}{\partial N} \ , \quad H^i = -\frac{\partial S}{\partial N_i} \ . \tag{29}$$

Expanding the Lagrangian (1) to second order about the FRW background (3) (with $N_i = 0$) leads to

$$\begin{aligned} P(X,\phi) \simeq P^{(0)} &+ \left(P^{(0)}_X \frac{\dot{\phi}_0\, \delta\dot{\phi}}{N^2} + P^{(0)}_\phi\, \delta\phi\right) \\ &+ \left(P^{(0)}_X + P^{(0)}_{XX} \frac{\dot{\phi}_0^2}{N^2}\right) \frac{\delta\dot{\phi}^2}{2\,N^2} + \frac{P^{(0)}_{\phi\phi}}{2}\,\delta\phi^2 \\ &- \frac{P^{(0)}_X}{2\,a^2}\,\delta\phi_{,i}^2 + P^{(0)}_{X\phi} \frac{\dot{\phi}_0\, \delta\dot{\phi}}{N^2}\,\delta\phi \ . \end{aligned} \tag{30}$$

Substituting the perturbed action in Eq. (29) (and setting $N = 1$) then leads to

$$H = H^{(0)} + \delta^{(1)} H + \delta^{(2)} H \ , \tag{31}$$

where

$$H^{(0)} = P^{(0)}_X \dot{\phi}_0^2 - P^{(0)} \tag{32a}$$

$$\begin{aligned} \delta^{(1)} H = &\left(P^{(0)}_X + P^{(0)}_{XX} \dot{\phi}_0^2\right) \dot{\phi}_0\, \delta\dot{\phi} \\ &- \left(P^{(0)}_\phi - P^{(0)}_{X\phi} \dot{\phi}_0^2\right) \delta\phi \end{aligned} \tag{32b}$$

$$\begin{aligned} \delta^{(2)} H = &\left(P^{(0)}_X + 4\,P^{(0)}_{XX} \dot{\phi}_0^2 + P^{(0)}_{XXX} \dot{\phi}_0^4\right) \frac{\delta\dot{\phi}^2}{2} \\ &+ \left(P^{(0)}_X - P^{(0)}_{XX} \dot{\phi}_0^2\right) \frac{\delta\phi_{,i}^2}{2\,a^2} - \left(P^{(0)}_{\phi\phi} - P^{(0)}_{X\phi\phi} \dot{\phi}_0^2\right) \frac{\delta\phi^2}{2} \\ &+ \left(P^{(0)}_{X\phi} + P^{(0)}_{XX\phi} \dot{\phi}_0^2\right) \dot{\phi}_0\, \delta\phi\, \delta\dot{\phi} \ . \end{aligned} \tag{32c}$$

Looking at (32c) we note the following:
i) at all orders, the super-Hamiltonian is identical to the 00-component of the stress tensor given in Eqs. (13), (14a) and (15a). Although this might seem obvious, we would like to point out that the two quantities are derived in different ways: the stress tensor is obtained from the complete matter action, whereas the super-Hamiltonian is obtained from the symmetry reduced (and perturbed) action for the matter alone. We will discuss more on this aspect below.
ii) like in the approach of the perturbed stress tensor, the *effective speed of propagation* of $\delta^{(2)} H$ defined in Eq. (4) is given by c_0 from Eq. (18a) and is identical to the speed of sound used in the literature [29] only under special conditions [satisfied by the canonical scalar field (16)].

V. SYMMETRY-REDUCED ACTION

Following the seminal works of Lukash [4], and Chibisov and Mukhanov [5], this procedure has been extensively used in quantifying primordial perturbations and their non-Gaussianity from inflation [19, 20]. The basic idea is to perturb the action about the FRW background, up to second (or higher) order, and reduce it so that the perturbations are described in terms of a single gauge-invariant variable, which will depend on the metric and matter content.

Here, our aim is to obtain the canonical Hamiltonian $\mathcal{H}$ corresponding to the perturbations of the generalized scalar field and compare with the quantities previously derived in the other approaches. Using the perturbed action up to second order from Eq. (30), and decomposing the modes in the Fourier domain, gives the following second-order action for the matter perturbations $\delta\phi_k$:

$$\begin{aligned} \delta^{(2)} S = \int {\rm d}t\, \frac{a^3}{2} \Big[&\left(P^{(0)}_X + P^{(0)}_{XX} \dot{\phi}_0^2\right) \delta\dot{\phi}_k^2 + 2\,P^{(0)}_{X\phi} \dot{\phi}_0\, \delta\phi_k\, \delta\dot{\phi}_k \\ &+ \left(P^{(0)}_{\phi\phi} - \frac{k^2}{a^2}\,P^{(0)}_X\right) \delta\phi_k^2 \Big] \ . \end{aligned} \tag{33}$$

Defining the canonical momentum conjugate to $\delta\phi_k$ as

$$P_k = \frac{\partial \delta^{(2)} S}{\partial \delta\dot{\phi}_k} \ , \tag{34}$$

the canonical Hamiltonian corresponding to the perturbed action (33) reads

$$\delta^{(2)}\mathcal{H} = \frac{a^3}{2} \left[\left(P^{(0)}_X + P^{(0)}_{XX} \dot{\phi}_0^2\right) \delta\dot{\phi}_k^2 + \left(\frac{k^2}{a^2}\,P^{(0)}_X - P^{(0)}_{\phi\phi}\right) \delta\phi_k^2\right] . \tag{35}$$

The point that needs to be emphasized here is that $\delta^{(2)}\mathcal{H}$ is identical to $\delta^{(2)} T_{00}$ and to the super-Hamiltonian $\delta^{(2)} H$ for the canonical scalar field (16), but differ for general non-canonical fields. [Note that $\delta^{(2)}\mathcal{H}$ is a Hamiltonian, while $\delta^{(2)} H$ is a Hamiltonian *density*. Hence, the expressions (35) and (32c) differ by an overall factor of a^3.]

Also, the ratio of the factors in front of $\delta\dot{\phi}_k^2$ and $k^2\,\delta\phi_k^2$ is equal to $c_\parallel^2$ given in Eq. (18b) and, thus, to the speed of sound (10) obtained from the equation of motion. This implies that $\delta^{(2)}T_{00}$ and the canonical Hamiltonian $\delta^{(2)}\mathcal{H}$ become unstable under different conditions.

The findings presented in this section are partly reminiscent of some general results found by Maccallum and Taub [34] (see also Sec. (13.2) in Ref. [35]), who showed that the variation of the action and the gauge fixing (in this case, the symmetry reduction) do not necessarily commute, hence the two procedures may not lead to the same equations of motion. To be precise, in Ref. [34], they found that variation and reduction should be commuting operations for Class A space-times, to which the FRW universe belongs, but their result was obtained assuming the presence of a standard fluid or a canonical scalar field. In this sense, our results can be considered as an extension of their studies to non-canonical Lagrangians, and show that the variation and gauge fixing do not commute even in a FRW universe when the scalar field is not canonical.

VI. DISCUSSION AND EXAMPLES

Let us now come to the main points of discussion in this note. The first question is why the canonical Hamiltonian, perturbed stress tensor and super-Hamiltonian coincide for a canonical scalar field, but not for general scalar field Lagrangians. To go about answering this question, it is necessary to look at the four approaches we have employed from a different perspective. In the first two approaches – perturbed stress tensor and covariant approach – we perturb the general expression for the stress tensor of the scalar field and obtain its second-order contribution $\delta^{(2)}T_{00}$. In the last two approaches – ADM formulation and symmetry-reduced action – we expand the action to second order in the perturbation and obtain the super-(canonical) Hamiltonian of the corresponding perturbed action. While the super-Hamiltonian $\delta^{(2)}H$ is identical to $\delta^{(2)}T_{00}$ [51], the canonical Hamiltonian $\delta^{(2)}\mathcal{H}$ is different.

This then raises a related question: Why is the super-Hamiltonian $\delta^{(2)}H$ different from the canonical Hamiltonian $\delta^{(2)}\mathcal{H}$ for non-canonical scalar fields? To answer this, let us assume that the coefficients containing the background quantities $P_X^{(0)}$ and $P_{XX}^{(0)}$ in the second-order action (30) are constant and *independent* of N. It is then easy to see that, using Eq. (29), the resulting super-Hamiltonian is identical to the canonical Hamiltonian (35) in this approximation. In other words, the canonical Hamiltonian given in Eq. (35) is consistent provided the time-variation of background quantities (like $P_X^{(0)}$ and $P_{XX}^{(0)}$) can be neglected. (For the canonical scalar field, these functions are indeed constant and the perturbed quantities therefore coincide.) Although such an approximation may be valid for specific non-canonical fields, they fail for some of the known fields used in the literature, as we now proceed to review.

A. k-essence

Let us consider the simplest non-canonical scalar field discussed in the context of power-law inflation,

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{2/3\gamma}\,, \qquad a_0 = a(t_0)\,, \tag{36}$$

whose Lagrangian is [27]

$$P = f(\phi)\left(X^2 - X\right)\,. \tag{37}$$

Upon solving the equation of state and the master equation for the evolution of the energy density ϵ (as derived from the Einstein field equations),

$$\epsilon + p = \gamma\,\epsilon\,, \qquad \dot{\epsilon} = -\sqrt{3\,\epsilon}\,\ell_{\rm p}\,(\epsilon + p)\,, \tag{38}$$

one finds

$$X^{(0)} = \frac{2-\gamma}{4-3\,\gamma}\,, \tag{39}$$

so that

$$\dot{\phi}_0 \equiv \sqrt{2\,X_0} = \sqrt{\frac{4-2\,\gamma}{4-3\,\gamma}} \tag{40}$$

is constant and the background scalar field evolves in time according to

$$\phi_0(t) = \sqrt{\frac{4-2\,\gamma}{4-3\,\gamma}}\,t\,. \tag{41}$$

One therefore finds that the background power-law "potential" also evolves in time, namely

$$\begin{aligned} f(\phi_0) &= \frac{4\,(4-3\,\gamma)\,f_0}{\left\{2\,\sqrt{4-3\,\gamma} + \sqrt{3\,f_0}\,\gamma\,\ell_{\rm p}\left[\phi_0(t)-\phi_0(t_0)\right]\right\}^2} \\ &= \frac{f_0}{\left[1+g_0\,(t-t_0)\right]^2}\,, \end{aligned} \tag{42}$$

where $f_0 \equiv f(\phi_0(t_0))$, and so evolve $P_X^{(0)}$ and $P_{XX}^{(0)}$.

In order to achieve an accelerated expansion, *i.e.* inflation, γ must range in $[0, 2/3]$ so that $X^{(0)}$ is inside the interval $[1/2, 2/3]$. The explicit expressions for the various "speeds of sound" are [see Eqs. (10), (18a) and (25a), respectively]

$$\begin{aligned} c_{\delta\phi}^2 &= \frac{2\,X^{(0)}-1}{6\,X^{(0)}-1} \\ c_0^2 &= -\frac{2\,X^{(0)}+1}{18\,X^{(0)}-1} \\ c_\rho^2 &= -\frac{6\,X^{(0)}-1}{18\,X^{(0)}-1}\,. \end{aligned} \tag{43}$$

On substituting Eq. (39) and taking into account the valid ranges for γ and $X^{(0)}$ given above, one finds

$$\begin{aligned} 0 < c_{\delta\phi}^2 < \tfrac{1}{9} \\ -\tfrac{1}{4} < c_0^2 < -\tfrac{7}{33} \\ -\tfrac{3}{11} < c_\rho^2 < -\tfrac{1}{4} \ . \end{aligned} \tag{44}$$

Hence, c_0 is imaginary (equivalently, $\delta^{(2)}T_{00}$ and the super-Hamiltonian $\delta^{(2)}H$ are unstable) for all values of $X^{(0)} > 1/2$ allowed by the background dynamics, and, in particular, for those required to achieve accelerated expansion. The same occurs for c_ρ (which is a decreasing function of $X^{(0)}$). However, as is well known, the velocity $c_{\delta\phi}$ is real and well-defined in the entire range of admissible γ and $X^{(0)}$. A possible physical interpretation of this finding is that, whereas the field perturbation $\delta\phi$ propagates with real and well defined velocity on the chosen background, its energy density grows in time and drives the system out of the perturbative regime. This result cannot be just a curiosity and the instability must have physical consequences, given that gravity necessarily couples to the energy density.

B. Tachyon

For the tachyon [36], whose Lagrangian is

$$P = -V(\phi)\sqrt{1-2X} \ , \tag{45}$$

where V is positive in the background FRW, the background dynamics requires that $X < 1/2$. If one further imposes that $\delta^{(2)}T_{00}$ (and the super-Hamiltonian) is stable (in the sense we already specified in the previous sections), one obtains the new constraint $X < 1/4$. Here the requirement that the perturbations must be stable leads to a smaller parameter range for the tachyonic field during inflation. If one instead uses Eq. (25a), one finds that $X > 1/2$ for c_ρ to be positive, which is incompatible with the all of the values allowed by the background dynamics mentioned above. Again, $c_{\delta\phi}$ is instead always real.

C. DBI field

We also find a similar situation for the Dirac-Born-Infeld (DBI) field [37, 38], whose Lagrangian is

$$P = -\frac{1}{f(\phi)}\left(\sqrt{1-2f(\phi)X} - 1\right) - V(\phi) \ , \tag{46}$$

with f and V positive functions in the background FRW space-time. The background dynamics requires that $X < 1/(2f_0)$ but, as above, if one further imposes that $\delta^{(2)}T_{00}$ (and the super-Hamiltonian) be stable, one obtains the stronger constraint $X < 1/(4f_0)$. Using Eq. (25a), one finds that $X > 1/(2f_0)$ for c_ρ to be positive. Just as it happens in the tachyonic case, this is incompatible with all of the values allowed by the background dynamics. Like in the previous examples, $c_{\delta\phi}$ is instead real and does not introduce new constraints.

VII. CONCLUSIONS

In this work, we have considered perturbations of a generalized scalar field in four different approaches. We have shown that second-order quantities obtained in these approaches coincide for the canonical scalar field but are in general different. At the root of the discrepancy lies the fact that, in evaluating the canonical Hamiltonian from the second-order action, one implicitly assumes that background quantities, like $P_X^{(0)}$ and $P_{XX}^{(0)}$, are constant. As appears clearly, *e.* g. for the k-essence reviewed in Section VI A, background quantities are in general time-dependent and neglecting this feature leads to incorrect expressions. In particular, one expects the faster the background evolves, the larger the discrepancy. This aspect should be taken into account, for example, when one applies adiabatic or slow-roll approximations.

We have also shown that instabilities in general occur in the components of the perturbed stress tensor to second order, signaled by imaginary *speeds of sound* (4). Consequently, for specific known non-canonical Lagrangians, demanding that the second-order stress tensor be stable against small field perturbations $\delta\phi$ restricts (and possibly rules out completely) the parameter range of the scalar field. Let us further recall that the energy density for the canonical scalar field in the background frame is always stable, but instabilities appear in the fluid-comoving frame, namely in the expressions for ρ and p. Our analysis indicates that, for instance, the results of Refs. [20, 39] should be carefully reanalyzed in other approaches to check for plausible inconsistencies. Also, it is important to repeat the analysis of Malik [40] – by considering second-order perturbations – for general scalar fields.

Our findings have been obtained by freezing the metric perturbations completely and one could naturally wonder if the instabilities we found would stand a more general investigation. Unfortunately, a complete treatment of metric perturbations to second order is extremely involved and goes beyond the scope of the present work. However, in Appendix C, we provide some preliminary results which suggest that metric perturbations should not affect the aforementioned instabilities, because they do not seem to contribute new terms to the ratios (4).

Our results naturally raise the question of *linearization instability* for non-canonical scalar fields [41]. It has been known in the literature that the linearization of non-linear fields can lead to spurious solutions [41–44]. In particular, Brill and Deser [42] showed that there are spurious solutions to the linearized Einstein's equations around the vacuum space-time given by the flat three-

torus with zero extrinsic curvature. In the case of perturbations of the FRW background, D'Eath [43] showed that the perturbations of Einstein's equations with the isentropic perfect fluid matter is *linearization stable*. (It is important to note that linearization instability is not directly related to other kinds of instability. However, dynamical instability is often studied by examining solutions of the linearized equations [44].) In Einstein gravity with canonical fields, linearization instability requires a compact space. However the non-canonical scalar fields are non-linear, which complicates the scenario. Our analysis shows that the perturbation of these fields about the spatially flat FRW background (by freezing the metric perturbations) might still turn out to be *linearization unstable*. It is thus important to repeat D'Eath's analysis for this class of scalar fields by including the metric perturbations about the FRW background. We hope to address this issue further in a future publication.

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Appendix A: Rescaled perturbation field

Terms with $\delta\dot\phi$ can be eliminated from the equation of motion (9a) by means of the rescaling (11) with

$$r(t) = \exp\left(-\frac{1}{2}\int^t \frac{C}{A}\,\mathrm{d}t'\right) \ , \tag{A1}$$

where C was given in Eq. (9b) and $A = P_X^{(0)} + P_{XX}^{(0)}\dot\phi_0^2$ is the coefficient of $\delta\ddot\phi$ in Eq. (9a). The equation for the rescaled field thus reads

$$A\,\ddot\chi + \frac{B}{a^2}\,\chi_{,ii} + \tilde D\,\chi = 0 \ , \tag{A2a}$$

where $B = -P_X^{(0)}$ and

$$\tilde D = D + \frac{C\,\dot A}{2\,A} - \frac{C^2}{4\,A} - \frac{\dot C}{2} \ . \tag{A2b}$$

This operation therefore modifies the coefficient of $\delta\phi$ (the "mass term"), leaving the speed of sound in the equation for χ the same as the one given in Eq. (10).

By similar treatments, terms linear in $\delta\dot\phi$ can be eliminated from the components of the stress tensor. For example, $\delta^{(2)}T_{00}$ has the following structure

$$\delta^{(2)}T_{00} = \bar A\,\delta\dot\phi^2 + \frac{\bar B}{a^2}\,\delta\phi_{,i}^2 + \bar C\,\delta\phi\,\delta\dot\phi + \bar D\,\delta\phi^2 \ , \tag{A3}$$

where the time-dependent coefficients $\bar A$, $\bar B$, $\bar C$ and $\bar D$ can be read from Eq. (15a). The rescaling in Eq. (11), now with

$$r(t) = \exp\left(-\frac{1}{2}\int^t \frac{\bar C}{\bar A}\,\mathrm{d}t'\right) \ , \tag{A4}$$

then yields

$$\delta^{(2)}T_{00} = \bar A\,\dot\chi^2 + \frac{\bar B}{a^2}\,\chi_{,i}^2 + \left(\bar D - \frac{\bar C^2}{4\,\bar A}\right)\chi^2 \ , \tag{A5}$$

or, in the Fourier domain,

$$\delta^{(2)}T_{00} = \bar A\,\dot\chi_k^2 + \left(\bar B\,\frac{k^2}{a^2} + \bar D - \frac{\bar C^2}{4\,\bar A}\right)\chi_k^2 \ . \tag{A6}$$

Hence, $\delta^{(2)}T_{00}$ is stable when

$$\frac{k^2}{a^2} \geq \frac{\bar C^2}{4\,\bar A\,\bar B} - \frac{\bar D}{\bar B} \ . \tag{A7}$$

Eq. (A7) can be regarded as a test that P (and its derivatives) must pass if one wants to deal with a model endowed with a stable perturbation theory. Similar conditions can be found for other quantities, like $\delta^{(2)}\rho$ in Eq. (22a) or $\delta^{(2)}p$ in Eq. (22b). Let us also note in passing that, when the inequality (A7) is saturated, the perturbation χ_k appears in a state of "asymptotic freedom" of the sort discussed in Ref. [47].

To summarise, although the transformation (11) with r given in Eq. (A1) or Eq. (A4) changes the values of the stress tensor, density and pressure, it does not affect the expressions of the corresponding velocities and cannot remove the instabilities signalled by imaginary speeds of propagation.

Appendix B: Equivalence between T_{00} and $\mathcal{H}$

The equivalence between T_{00} and the (matter part of the) super-Hamiltonian $\mathcal{H}$ can be proven in general in an FRW space-time in the proper time gauge. From the standard definition for T_{00},

$$T_{00} = \frac{2}{\sqrt{-g}}\,\frac{\delta S}{\delta g^{00}} \ , \tag{B1}$$

we find, for the metric (3),

$$T_{00} = -\frac{N^2}{a^3}\,\frac{\delta S}{\delta N} \equiv \frac{N^2}{a^3}\,H \ . \tag{B2}$$

The energy contained in a given spatial (comoving) volume V is equal to

$$E = \int_V \mathrm{d}^3x\,\sqrt{-g}\,T_{00} = \int_V \mathrm{d}^3x\,N^3\,H = V\,N^3\,H \ , \tag{B3}$$

where V can be set equal to one without loss of generality. It follows that, in the proper time gauge ($N = 1$), one has $E = H$, namely the super-Hamiltonian always equals the energy whose density is given by T_{00}.

Appendix C: Metric perturbations: preliminary results

A complete description of metric perturbations to second order remains outside the scope of the present work and we just outline the general features one might encounter by including the metric perturbations in two approaches: the perturbed stress tensor and reduced action.

For the general Lagrangian (1), the stress tensor is given by Eq. (6). The general perturbation about the FRW background, *i. e.*,

$$\begin{aligned} \phi &= \phi_{_0}(t) + \epsilon\, \delta\phi(t,\mathbf{x}) \\ g_{\mu\nu} &= g^{(0)}_{\mu\nu}(t) + \epsilon\, \delta g_{\mu\nu}(t,\mathbf{x}) \ , \end{aligned} \tag{C1}$$

leads to

$$\begin{aligned} T_{_{\mu\nu}} = T_{\mu\nu}|_0 &+ \epsilon \left[\left.\frac{\partial T_{\mu\nu}}{\partial X}\right|_0 \delta X + \left.\frac{\partial T_{\mu\nu}}{\partial \phi}\right|_0 \delta\phi + \left.\frac{\partial T_{\mu\nu}}{\partial g_{\alpha\beta}}\right|_0 \delta g_{\alpha\beta} \right] \\ &+ \frac{\epsilon^2}{2} \left[\left.\frac{\partial^2 T_{\mu\nu}}{\partial X^2}\right|_0 (\delta X)^2 + \left.\frac{\partial^2 T_{\mu\nu}}{\partial \phi^2}\right|_0 (\delta\phi)^2 \right. \\ &\left. + \left.\frac{\partial^2 T_{\mu\nu}}{\partial g^2_{\alpha\beta}}\right|_0 (\delta g_{\alpha\beta})^2 \right] + \text{ cross terms} \ , \end{aligned} \tag{C2}$$

where $|_0$ means the expression is evaluated on the background quantities ϕ_0, X_0 and $g^{(0)}_{\alpha\beta}$. Focusing on the metric perturbations, we have

$$\begin{aligned} \frac{\partial T_{\mu\nu}}{\partial g_{\alpha\beta}} &= \frac{\partial^\alpha\phi\, \partial^\beta\phi}{2} \left(P_{_{XX}}\, \partial_\mu\phi\, \partial_\nu\phi - g_{\mu\nu}\, P_{_X} \right) - \delta^\alpha_\mu\, \delta^\beta_\nu\, P \\ \frac{\partial^2 T_{\mu\nu}}{\partial g_{\gamma\delta}\, \partial g_{\alpha\beta}} &= \frac{\partial^\alpha\phi\, \partial^\beta\phi\, \partial^\gamma\phi\, \partial^\delta\phi}{4} \\ &\quad \times \left(P_{_{XXX}}\, \partial_\mu\phi\, \partial_\nu\phi - g_{\mu\nu}\, P_{_{XX}} \right) \\ &\quad - \frac{P_{_X}}{2} \left(\delta^\alpha_\mu\, \delta^\beta_\nu\, \partial^\gamma\phi\, \partial^\delta\phi + \delta^\gamma_\mu\, \delta^\delta_\nu\, \partial^\alpha\phi\, \partial^\beta\phi \right) \ . \end{aligned} \tag{C3}$$

Evaluating the above expressions in the FRW background, we get

$$\begin{aligned} \left.\frac{\partial T_{\mu\nu}}{\partial g_{\alpha\beta}}\right|_{_{\rm FRW}} &= \frac{\delta^\alpha_0\, \delta^\beta_0\, \dot\phi^2_{_0}}{2} \left(P^{^{(0)}}_{_{XX}}\, \dot\phi^2_{_0} \delta^0_\mu\, \delta^0_\nu - g^{(0)}_{\mu\nu}\, P^{^{(0)}}_{_X} \right) \\ &\quad - \delta^\alpha_\mu\, \delta^\beta_\nu\, P^{^{(0)}} \\ \left.\frac{\partial^2 T_{\mu\nu}}{\partial g_{\gamma\delta}\, \partial g_{\alpha\beta}}\right|_{_{\rm FRW}} &= \frac{\delta^\alpha_0\, \delta^\beta_0\, \delta^\gamma_0\, \delta^\delta_0\, \dot\phi^4_{_0}}{4} \\ &\quad \times \left(P^{^{(0)}}_{_{XXX}}\, \dot\phi^2_{_0}\, \delta^0_\mu\, \delta^0_\nu - g^{(0)}_{\mu\nu}\, P^{^{(0)}}_{_{XX}} \right) \\ &\quad - \frac{P^{^{(0)}}_{_X}}{2} \left(\delta^\alpha_\mu\, \delta^\beta_\nu\, \delta^\gamma_0\, \delta^\delta_0 + \delta^\gamma_\mu\, \delta^\delta_\nu\, \delta^\alpha_0\, \delta^\beta_0 \right) \dot\phi^2_{_0} \ . \end{aligned} \tag{C4}$$

For the 00-component of the stress tensor, we then have

$$\begin{aligned} \left.\frac{\partial T_{00}}{\partial g_{\alpha\beta}}\right|_{_{\rm FRW}} &= \frac{\delta^\alpha_0\, \delta^\beta_0\, \dot\phi^2_{_0}}{2} \left(P^{^{(0)}}_{_{XX}}\, \dot\phi^2_{_0} - a^2\, P^{^{(0)}}_{_X} \right) - \delta^\alpha_0\, \delta^\beta_0\, P^{^{(0)}} \\ \left.\frac{\partial^2 T_{00}}{\partial g_{\gamma\delta}\, \partial g_{\alpha\beta}}\right|_{_{\rm FRW}} &= \frac{\delta^\alpha_0\, \delta^\beta_0\, \delta^\gamma_0\, \delta^\delta_0}{4} \\ &\quad \times \left(P^{^{(0)}}_{_{XXX}}\, \dot\phi^6_{_0} - a^2\, P^{^{(0)}}_{_{XX}}\, \dot\phi^4_{_0} - 4\, P^{^{(0)}}_{_X}\, \dot\phi^2_{_0} \right) \ . \end{aligned} \tag{C5}$$

By comparing the second-order expression of the stress tensor (15a) with Eq. (C5), we immediately see a peculiar difference: $P^{^{(0)}}_{_{XXX}}$ in Eq. (C5) multiplies $\dot\phi^6_{_0}$, whereas it multiplies $\dot\phi^4_{_0}$ in Eq. (15a). This means that metric perturbations and matter perturbations give rise to different contributions to the stress tensor and including the metric perturbations should not, in principle, remove the instability we found at second order. Moreover, Eq. (C5) makes it clear that investigating the complete second-order perturbations is highly non-trivial.

We finally look at the metric perturbations in the reduced action approach. On substiuting (C1) in Eq. (1), the Lagrangian can be written as

$$\begin{aligned} P = P^{^{(0)}} &+ \epsilon \left[\left.\frac{\partial P}{\partial X}\right|_0 \delta X + \left.\frac{\partial P}{\partial \phi}\right|_0 \delta\phi + \left.\frac{\partial P}{\partial g_{\mu\nu}}\right|_0 \delta g_{\mu\nu} \right] \\ &+ \frac{\epsilon^2}{2} \left[\left.\frac{\partial^2 P}{\partial X^2}\right|_0 (\delta X)^2 + \left.\frac{\partial^2 P}{\partial \phi^2}\right|_0 (\delta\phi)^2 \right. \\ &\left. + \left.\frac{\partial^2 P}{\partial g_{\gamma\delta}\, \partial g_{\alpha\beta}}\right|_0 \delta g_{\alpha\beta}\, \delta g_{\gamma\delta} \right] + \text{ cross terms} \ . \end{aligned} \tag{C6}$$

Again, focusing on the metric perturbations, we have

$$\begin{aligned} \frac{\partial P}{\partial g_{\alpha\beta}} &= \frac{P_{_X}}{2}\, \partial^\alpha\phi\, \partial^\beta\phi \\ \frac{\partial^2 P}{\partial g_{\alpha\beta}\, \partial g_{\gamma\delta}} &= \frac{1}{4}\, \partial^\alpha\phi\, \partial^\beta\phi\, \partial^\gamma\phi\, \partial^\delta\phi\, P_{_{XX}} \ . \end{aligned} \tag{C7}$$

The above expressions in the FRW background yield

$$\begin{aligned} \left.\frac{\partial P}{\partial g_{\mu\nu}}\right|_{_{\rm FRW}} &= \frac{P_{_X}}{2}\, \delta^\mu_0\, \delta^\nu_0\, \dot\phi^2_{_0} \\ \left.\frac{\partial^2 P}{\partial g_{\alpha\beta}\, \partial g_{\gamma\delta}}\right|_{_{\rm FRW}} &= \frac{1}{4}\, \delta^\alpha_0\, \delta^\beta_0\, \delta^\gamma_0 \delta^\delta_0\, \dot\phi^4_{_0}\, P_{_{XX}} \ . \end{aligned} \tag{C8}$$

Comparing the above expressions with Eq. (23b), we note that the contributions of the metric perturbations to the second-order action differ from the contributions of the metric perturbations to the second-order stress tensor.

From the above preliminary analysis, it seems that the features we have obtained in this work continue to hold with the metric perturbations. However, this issue needs a thorough investigation and will be discussed elsewhere.

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[48] Our analysis is similar in spirit to Gruzinov's calculation [39].
[49] See, for instance, Sec. (39.7) in Ref. [45].
[50] For example, in Ref. [46] (Sec. 8.3, page 276), $\delta^{(2)}\rho$ is claimed to be positive, but no proof is given from first principles. We thank D. Wands for pointing this out.
[51] In Appendix B, we provide a general proof that the stress tensor and super-Hamiltonian are equal to all orders in the FRW background.